NONSTATIONARY MAGNETOHYDRODYNAMIC COUETTE FLOW OF A CONDUCTING LIQUID AT A CONSTANT RATE

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An approximate solution of the problem is presented for the case of laminary nonstationary Couette flow due to the change in the applied magnetic field at a constant rate of flow of a viscous incompressible conducting liquid.

We consider the nonstationary flow of a viscous incompressible conducting liquid between two parallel flat walls, one of which (x = 0) is an insulator and moves with constant velocity V, and the other (x = b) is a conductor and at rest (Fig. 1). The nonstationary character of the flow is due to an abrupt change in the value of the homogeneous magnetic field (turning on, turning off, or regulation of the induction of the magnetic field), which is perpendicular to both walls. We assume that the rate of liquid flow is maintained constant. In this case, unlike in [1], the longitudinal pressure gradient is not equal to zero and is a function of the time.

Under these conditions and at a small magnetic Reynolds number, the system of magnetohydrodynamics equations can be reduced to a single equation for the velocity of the liquid

$$\frac{\partial v}{\partial t} = P(t) - \frac{M^2}{b^2}\nu v + \nu\frac{\partial^2 v}{\partial x^2} \qquad \left(P(t) = -\frac{1}{\rho}\frac{\partial p}{\partial z}\right) \tag{1}$$

$$\nu = \eta/\rho, \qquad M = bB\sqrt{\sigma/\eta}$$

Here v is the velocity of the liquid flow, ν is the kinematic viscosity coefficient, and M is the Hartmann number.

Integrating Eq. (1) over the height of the channel and using the condition that the flow rate is constant, we obtain

$$P(t) = \frac{M^2}{b^2}\nu V_f - \frac{\nu}{b}\int_0^b \frac{\partial^2 v}{\partial x^2}dx, \qquad V_f = \frac{1}{b}\int_0^b v\,dx \tag{2}$$

Here V_f is the average velocity of the liquid flow.

Eliminating the function P(t) with the aid of (2), we obtain

$$\frac{\partial v}{\partial t} = \frac{M^2\nu}{b^2}(V_f - v) - \frac{\nu}{b}\int_0^b \frac{\partial^2 v}{\partial x^2}dx + \nu\frac{\partial^2 v}{\partial x^2} \tag{3}$$

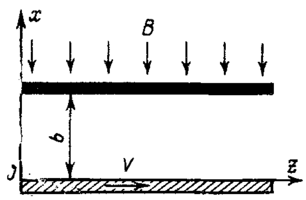

Fig. 1

The boundary conditions for Eq. (3) are

$$v = V, \quad x = 0, \quad v = 0, \quad x = b \tag{4}$$

The initial conditions are specified in the form

$$\left.\begin{array}{l} M = M_0 \\ v = v_0(x, 0) \end{array}\right\} \text{when } t = 0, \qquad \begin{array}{l} V_f = \text{const} \\ V = \text{const} \end{array} \tag{5}$$

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We seek the general solution of (3) in the form

$$v(x, t) = F(x) + \sum_{n=1}^{\infty} \Phi_n(t) \psi_n(x)$$

Here the function F(x) corresponds to the asymptotic value of the liquid flow rate.

According to the Fourier method, the initial equation (3) breaks up into a system of three differential equations

$$\begin{aligned} &\frac{d^2F(x)}{dx^2} - \frac{M^2}{b^2} F(x) + \frac{M^2}{b^2} V_f - \frac{1}{b}\int_0^b \frac{d^2F(x)}{dx^2}\,dx = 0 \\ &\frac{1}{\Phi_n(t)} \frac{d\Phi_n(t)}{dt} = -\alpha_n^2 \qquad \left(\alpha_n^2 = \nu\left(\beta_n + \frac{M^2}{b^2}\right)\right) \\ &\frac{1}{\psi_n(x)} \frac{d^2\psi_n(x)}{dx^2} - \frac{1}{b\psi_n(x)} \int_0^b \frac{d^2\psi_n(x)}{dx^2}\,dx = -\beta_n \end{aligned} \tag{6}$$

The eigenvalues β_n are determined here from the following transcendental equation:

$$\mathrm{tg}\,{}^1/_2 b \sqrt{\beta_n} = {}^1/_2 b \sqrt{\beta_n} \tag{7}$$

The solution of the first equation of the system (6), with allowance for the boundary conditions (4), is

$$F(x) = V\left\{H(M)\left(1 - \mathrm{ch}\,\frac{Mx}{b} + \mathrm{sh}\,\frac{Mx}{b}\,\mathrm{th}\,\frac{M}{2}\right) + \frac{\mathrm{sh}\,M(1 - x/b)}{\mathrm{sh}\,M}\right\} \tag{8}$$

$$H(M) = \frac{\mathrm{th}\,{}^1/_2 M - MV_f/V}{2\,\mathrm{th}\,{}^1/_2 M - M} \tag{9}$$

The general solutions of the second and third equations of the system (6) are, respectively,

$$\Phi_n(t) = C_{2n} \exp(-\alpha_n^2 t), \qquad \psi_n(x) = C_{3n}\left({}^1/_2 b \sqrt{\beta_n} \sin \sqrt{\beta_n}\,x + \cos \sqrt{\beta_n}\,x - 1\right) \tag{10}$$

Here C_{2n} and C_{3n} are arbitrary constants determined from the initial conditions (5) and the boundary conditions (4).

Thus, the general solution of the equation of motion of the liquid (3) is

$$v(x, t) = V\left\{H(M)\left(1 - \mathrm{ch}\,\frac{Mx}{b} + \mathrm{sh}\,\frac{Mx}{b}\,\mathrm{th}\,\frac{M}{2}\right) + \frac{\mathrm{sh}\,M(1 - x/b)}{\mathrm{sh}\,M}\right\} + \sum_{n=1}^{\infty} C_n\left[{}^1/_2 b\sqrt{\beta_n} \sin\sqrt{\beta_n}\,x + \cos\sqrt{\beta_n}\,x - 1\right]\exp(-\alpha_n^2 t) \tag{11}$$

To determine the coefficients of the series C_n, it is necessary to establish the orthogonality of the eigenfunctions $\psi_n(x)$, i.e., to satisfy the condition

$$\int_0^b \psi_n(x)\,\psi_m(x)\,dx = 0 \text{ when } n \neq m$$

In this case this condition is satisfied. Then, using the initial condition for the velocity of the liquid

$$v_0(x, 0) = V\left\{H(M_0)\left(1 - \mathrm{ch}\,\frac{M_0 x}{b} + \mathrm{sh}\,\frac{M_0 x}{b}\,\mathrm{th}\,\frac{M_0}{2}\right) + \frac{\mathrm{sh}\,M_0(1 - x/b)}{\mathrm{sh}\,M_0}\right\}$$

we obtain an expression for C_n in the form

$$C_n = \frac{V}{J}\{H(M_0)J_1 - H(M)J_2 - J_3 - J_4\} \tag{12}$$

Here H(M) is defined by (9),

$$J_1 = \frac{b}{M_0^2 + b^2\beta_n}\left\{\frac{M_0^2(4 + b^2\beta_n)}{4\sqrt{\beta_n}\,b} \sin\sqrt{\beta_n}\,b + \frac{2b^2\beta_n}{M_0}\,\mathrm{th}\,\frac{M_0}{2}\right\} - b$$

$$J_2 = \frac{b}{M^2 + b^2\beta_n}\left\{\frac{M^2(4 + b^2\beta_n)}{4\sqrt{\beta_n}\,b}\sin\sqrt{\beta_n}\,b + \frac{2b^2\beta_n}{M}\operatorname{th}\frac{M}{2}\right\} - b$$

$$J_3 = \frac{b^3\beta_n(M - 2\operatorname{th}{}^1/_2M)}{2M(M^2 + \beta_n b^2)}, \qquad J_4 = \frac{b^3\beta_n(M_0 - 2\operatorname{th}{}^1/_2M_0)}{2M_0(M_0^2 + \beta_n b^2)} \qquad (13)$$

$$J = \int_0^b \psi_n^2(x)\,dx = b\left\{0.75 + \frac{b^2\beta_n}{8} + \frac{16 + 8b^2\beta_n + b^4\beta_n^4}{32\sqrt{\beta_n}\,b(4 - b^2\beta_n)}\sin 2\sqrt{\beta_n}\,b - \frac{b^2\beta_n + 4}{4b\sqrt{\beta_n}}\sin\sqrt{\beta_n}\,b\right\}$$

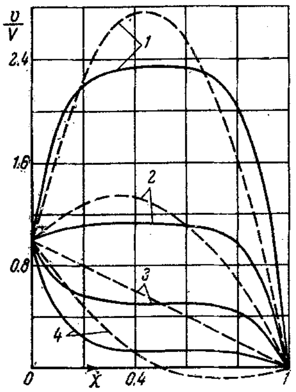

Fig. 2

If the nonstationary liquid flow is due to turning on the magnetic field under the initial conditions

$$t = 0, \quad M_0 = 0, \quad v_0(x, 0) = V\left\{\left(\frac{3x}{b} - 1\right) - \frac{V_f}{V}\frac{6x}{b}\right\}\left(\frac{x}{b} - 1\right)$$

then the coefficients of the series are given by

$$C_n = \frac{V}{J}\left\{\frac{b^2\beta_n + 2}{2b\beta_n} - \frac{1}{2}\left(\frac{b}{4} + \frac{1}{b\beta_n}\right)\cos\sqrt{\beta_n}\,b - \frac{V_f}{V}b - H(M)J_2 - J_3\right\} \qquad (14)$$

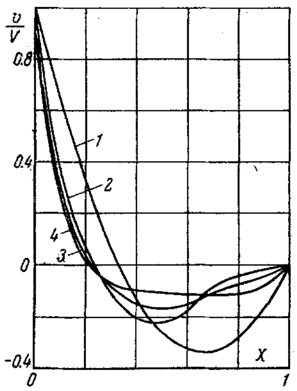

Fig. 3

Here J is defined by (13) and H(M) by (9).

The derived relations (11)-(14) were used to calculate the Couette flow. Figure 2 shows diagrams of the liquid velocity under steady-state flow in ordinary hydrodynamics (dashed lines) and for a magnetohydrodynamic stream with M = 10 (continuous lines) as functions of the dimensionless coordinate x/b = X. Curves 1, 2, 3, 4, and 5 correspond to V_f/V = 2.0, 1.0, 0.5, and 0.2.

We see that in the steady state, in the presence of a magnetic field, the velocity gradient of the liquid on the immobile wall of the channel increases relative to the usual hydrodynamic flow at all values of V_f/V, whereas, on the moving wall, it can either increase or decrease, depending on the ratio V_f/V.

Figure 3 shows diagrams of the liquid velocity at different instants of the nonstationary process due to a sudden turning on of the magnetic field, for the case when $V_f/V = 0$, $M_0 = 0$, and $M = 10$. The dimensionless time of the transient is assumed to be the quantity $\theta = t\nu/b^2$. Curves 1, 2, 3, and 4 correspond to θ = 0.0001, 0.005, and infinity at $M_0 = 0$.

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